

# RIEMANNIAN GEOMETRY AND THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. If a (non-constant) polynomial has no zero, then a certain Riemannian metric is constructed on the two dimensional sphere. Several geometric arguments are then shown to contradict this fact.

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## 1. INTRODUCTION

In [1] the authors proved that the Gauss-Bonnet theorem implies the fundamental theorem of algebra. In this note we present several new Riemannian geometry arguments which lead also to the fundamental theorem of algebra. All the proofs are based on the following technical result:

**Lemma 1.** *If there exists an irreducible polynomial  $p(z)$  of degree  $n > 1$ , then there exists a Riemannian metric  $g$  on the sphere  $\mathbb{S}^2$  such that its Gaussian curvature,  $K_g$ , vanishes identically.*

This result was already proved in [1] but we state it here in a completely different approach that, in our opinion, is more systematic than the previous one.

Clearly, the metric stated in Lemma 1, if it exists, it is a quite strange geometric object. Indeed, the second step in all the proofs we present here consist of showing that this metric cannot exist. In other words, we will point out several well-known geometrical obstructions to the construction of a flat metric on the sphere  $\mathbb{S}^2$ .

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In Section 2 we prove Lemma 1. Section 3 is devoted to explain the distinct arguments leading to a proof of the fact that sphere is not flat. Finally, in Section 4 we connect our proof to the field extension version of the fundamental theorem of algebra.

## 2. PROOF OF LEMMA 1

Assume  $p(z)$  is an irreducible polynomial of degree  $n > 1$ . This implies that the quotient  $A := \mathbb{C}[z] / \langle p(z) \rangle$  is a field. Furthermore, the map  $\tau : \mathbb{C}^n \longrightarrow A$  given by

$$\tau(a_0, \dots, a_{n-1}) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + \langle p(z) \rangle$$

defines an isomorphism of complex vector spaces. In particular,

$$\beta = \{ \tau(0, \dots, 1^{\text{i-th position}}, \dots, 0) \}_{i=1}^n$$

is a basis of  $A$ . Moreover, we have that  $\tau(-w, 1, 0, \dots, 0), \tau(-1, w, 0, \dots, 0) \neq 0$ , for any  $w \in \mathbb{C}$ . Hence

$$H(w) = \tau(-w, 1, 0, \dots, 0) \tau(-1, w, 0, \dots, 0) \neq 0.$$

Let  $M(w)$  be the associated matrix, with respect to the basis  $\beta$  above, to the complex linear operator  $L_w : A \longrightarrow A$  given by

$$L_w(\tau(a_0, \dots, a_{n-1})) = H(w) \tau(a_0, \dots, a_{n-1}).$$

Obviously,  $L_w$  is an isomorphism since  $H(w) \neq 0$  and  $A$  is a field. Hence  $\det(M(w)) \neq 0$  for all  $w$ . Furthermore,  $f(w) := \det(M(w))$  is a polynomial.

Now, the linearity of  $\tau$  guarantees that, for all  $w \in \mathbb{C} \setminus \{0\}$ ,

$$\begin{aligned} H(1/w) &= \tau(-1/w, 1, 0, \dots, 0) \tau(-1, 1/w, 0, \dots, 0) \\ &= [(1/w) \tau(-1, w, 0, \dots, 0)] [(1/w) \tau(-w, 1, 0, \dots, 0)] \\ &= (1/w^2) H(w), \end{aligned}$$

so that

$$f(1/w) = \det(M(1/w)) = \det((1/w^2)M(w)) = (1/w^{2n}) \det(M(w)) = (1/w^{2n})f(w).$$

It follows that there exists a Riemannian metric  $g$  on the sphere  $\mathbb{S}^2 = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , such that

$$g = \frac{1}{|f(w)|^{\frac{2}{n}}} |dw|^2 \quad \text{for } w \in \mathbb{C}$$

and

$$g = \frac{1}{|f(1/w)|^{\frac{2}{n}}} |d(1/w)|^2 \quad \text{for } w \in \widehat{\mathbb{C}} \setminus \{0\}.$$

Now, a simple computation shows that the Gaussian curvature  $K_g$  of  $g$  satisfies

$$\frac{1}{|f(w)|^{\frac{1}{n}}} K_g = \frac{1}{n} \Delta \log |f(w)| = \frac{1}{n} \Delta \operatorname{Re} \log f(w) = 0 \quad \text{for all } w \in \mathbb{C} \setminus \{0\},$$

since the real part of a holomorphic function must be harmonic. This obviously implies that  $K_g = 0$  on the whole sphere and ends the proof.

### 3. THE SPHERE IS NOT FLAT

Of course, the following arguments leading the title of this section are well-known. We recall them for sake of completeness of this note.

**First argument.** Any Riemannian metric on  $\mathbb{S}^2$  must be geodesically complete, from the Hopf-Rinow theorem. Therefore, the flat Riemannian manifold  $(\mathbb{S}^2, g)$  is also geodesically complete, and, taking into account that  $\mathbb{S}^2$  is connected and simply connected, the Cartan theorem on the classification of space forms (see [2, Theorem 7.10], for instance) gives that  $(\mathbb{S}^2, g)$  should be globally isometric to Euclidean plane, which is impossible because of the compactness of the sphere.

**Second argument.** The usual Riemannian metric of Gaussian curvature 1 on  $\mathbb{S}^2$  is locally written  $g^0 = (4/(1 + |w|^2)^2) |dw|^2$ ,  $w \in \mathbb{C}$ . Therefore, the Riemannian metric  $g$  in Lemma 1 is pointwise conformally related to  $g^0$ , i.e.,  $g = e^{2u} g^0$ , where  $u \in C^\infty(\mathbb{S}^2)$  is non constant (note that a homothetical metric to  $g^0$  has constant positive Gaussian curvature). Using now the relation between the Gaussian curvatures of two pointwise conformally related metrics we get  $\Delta^0 u = 1$ , where  $\Delta^0$  is the Laplacian relative to the metric  $g^0$ . Making use again of the compactness of the sphere, the classical maximum principle gets that  $u$  must be constant which is impossible.

**Third argument** [1]. As an easy consequence of the Gauss-Bonnet theorem, any Riemannian metric on  $\mathbb{S}^2$  has some elliptic point, i.e., a point where its Gaussian curvature is strictly positive. Hence, the existence of the metric  $g$  in Lemma 1 contradicts the Gauss-Bonnet theorem.

**Remark 1.** It should be noted that is crucial for our purposes that the the Gaussian curvature of the metric  $g$  in Lemma 1 is zero on all  $\mathbb{S}^2$ . Riemannian metrics on a sphere with non-constant Gaussian curvature  $K$  such that  $0 \leq K \leq 1$  and  $K = 0$  on a non zero measure set are known to exist.

## 4. A FINAL COMMENT

The proof of Lemma 1 we have presented in this note can be adapted with no extra effort to give a proof of the fact:

If  $A$  is a commutative  $\mathbb{C}$ -algebra,  $M$  is a maximal ideal of  $A$  and  $x + M \in A/M$  is algebraic of degree  $n > 1$  over  $\mathbb{C}$ , then there exists a Riemannian metric  $g$  on the sphere  $\mathbb{S}^2$  such that  $K_g$  vanishes identically.

This, in conjunction with the arguments in previous section, leads to a new and direct proof of the following well-known result:

**Theorem 2** (Field extension version of FTA). *Let  $A$  be a commutative  $\mathbb{C}$ -algebra and let  $M$  be a maximal ideal of  $A$ . If  $A/M$  is an algebraic field extension of  $\mathbb{C}$  (in particular, if  $[A/M : \mathbb{C}] = \dim_{\mathbb{C}}(A/M) = n < \infty$ , where  $\dim_{\mathbb{C}} V$  denotes complex dimension) then  $[A/M : \mathbb{C}] = 1$ .*

## REFERENCES

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